International Journal of Theoretical Physics, Vol. 39, No. 3, 2000

A Hilbert Space Realization of Nonlinear Quantum Mechanics as Classical Extension of Its Linear Counterpart†

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Received December 8, 1999

This paper studies the state-effect-probability structure associated with the quantum mechanics of nonlinear (homogeneous, in general nonadditive) operators on a Hilbert space. Its aim is twofold: to provide a concrete representation of the features of nonlinear quantum mechanics on a Hilbert space, and to show that the properties of the nonlinear version of quantum mechanics here described have the structure of a *classical* logic.

This paper discusses the concrete structure of nonlinear (homogeneous, in general nonadditive) operators on a Hilbert space as a possible nonlinear version of standard quantum mechanics. The role of nonlinear structures has been the subject of considerable discussion for some years. As well as the work of Beltrametti and Bugajski [1–4] and Bugajski [5], there have also been attempts to incorporate nonlinear operators in quantum mechanics by, among others, Mielnik [12, 14, 15], Haag and Bannier [11], and Weinberg [18, 17]. This paper describes a Hilbert space representation of such a nonlinear theory and shows how the logic associated with the properties of this representation is classical.

Section 1 provides a brief introduction to the standard state and effect of quantum mechanics. Section 2 briefly recapitulates the concrete Hilbert space structure normally identified as the standard linear quantum mechanics;

† This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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Section 3 shows how the nonlinear homogeneous structure arises, and proves that the logic of this structure is classical.

1. THE AXIOMATIC APPROACH TO UNSHARP QUANTUM MECHANICS

1.1. The Structure of State–Question–Probability in Axiomatic Quantum Mechanics

A structure of state-question-probability is a triple (S, Q, P), where:

1. S is a nonempty set, the elements of which are called *states*. In state $x \in S$, both individual samples and ensembles of identical noninteracting physical systems are prepared under well-defined and repeatable conditions by a macroscopic apparatus.
2. Q is a nonempty set, the elements of which are called *questions*. Question $q \in Q$ is tested by a dichotomic measuring macroscopic device which produces a certain definite macroscopic alternative when interacting with a single sample of the physical entity. The occurrence of the alternative is taken as the answer "yes" and its absence as the answer "no."
3. $P: S \times Q \to [0, 1]$ is a function, called the *probability function*. The value $P(x, q)$ represents the probability of the question q occurring relative to the state x.

Every question $q \in Q$ determines the following subsets of S:

- The *certainly-yes domain* of q: $S_1(q) := \{x \in S: P(x, q) = 1\}$.
- The *certainly-no domain* of q: $S_0(q) := \{x \in S: P(x, q) = 0\}$.

The triple will satisfy the following axioms:

Axiom 1. The principle of *indistinguishability* of states: two states x_1, x_2 are identical iff they produce the same statistical distribution. Formally,

$$\text{If for every } q \in Q, \quad P(x_1, q) = P(x_2, q), \quad \text{then } x_1 = x_2.$$

Axiom 2. The existence of the *certain* and *impossible* questions:

$$\exists \mathbb{I} \in Q: \quad \forall x \in S, \quad P(x, \mathbb{I}) = 1$$

$$\text{and} \qquad \exists \mathbb{O} \in Q: \quad \forall x \in S, \quad P(x, \mathbb{O}) = 0$$

Axiom 3. The existence of the *inverse question* of every question:

$$\forall q \in Q, \quad \exists q' \in Q: \quad \forall x \in S, \quad P(x, q) + P(x, q') = 1$$

Axiom 4. The existence of the *partial sum operation* defined for every pair of mutually orthogonal questions. Let p, q be two questions; then the *orthogonality condition* $\forall x \in S$, $P(x, p) + P(x, q) \leq 1$ implies that

$$\exists p \oplus q \in Q: \quad \forall x \in S, \quad P(x, p \oplus q) = P(x, p) + P(x, q)$$

Axiom 5. The existence of the *convex product* operation:

$$\forall q \in Q, \quad \forall \lambda \in [0, 1], \quad \exists (\lambda \cdot q) \in Q: \quad \forall x \in S, \quad P(x, \lambda \cdot q) = \lambda P(x, q)$$

Axiom 6. The existence of the *necessary question* of every question: $\forall q \in Q$, $\exists q^{\nu} \in Q$:

1. $S_1(q^{\nu}) = S_1(q)$
2. if $p \in Q$ satisfies $S_1(p) = S_1(q)$, then $\forall x \in S$, $P(x, q^{\nu}) \leq P(x, p)$.
3. if $r \in Q$ satisfies $S_0(r) = S_0(q^{\nu})$, then $\forall x \in S$, $P(x, r) \leq P(x, q^{\nu})$.

Assuming these axioms, the binary relation on Q defined as

$$p \preccurlyeq q \Leftrightarrow_{def} \forall x \in S, \quad P(p, x) \leq P(q, x) \tag{1.1}$$

is a partial quasiorder relation, i.e., the following hold:

(qo-1) $\forall q \in Q$, $q \preccurlyeq q$ [reflexivity]

(qo-2) $\forall p, q, r \in Q$, $p \preccurlyeq q$, and $q \preccurlyeq r$ imply $p \preccurlyeq r$ [transitivity]

(o-3) $\forall q \in Q$, $\mathbb{O} \preccurlyeq q \preccurlyeq \mathbb{I}$

Note that from these axioms it does not follow that the relation $\preccurlyeq$ is also antisymmetric.

1.2. The Structure of State–Effect–Probability

The relation (1.1) is also called the *physical quasiorder* relation. We recall that the condition $\forall x \in S$, $P(x, p) = P(x,q)$ [i.e., $p \preccurlyeq q$ and $q \preccurlyeq p$] does not imply the equality of the two questions: $p = q$. Thus, it is possible to introduce the following equivalence relation of *physical indistinguishability* of questions:

$$p \equiv_s q \Leftrightarrow_{def} \forall x \in S, \quad P(x, p) = P(x, q) \tag{1.2}$$

For every question $q \in Q$, let $[q] := \{p \in Q: p \equiv_s q\}$ be the equivalence class generated by q, modulo $\equiv_s$. Clearly $[\mathbb{O}] = \{q_0 \in Q: \forall x \in S, P(x, q_0) = 0\}$ and $[\mathbb{I}] = \{q_1 \in Q: \forall x \in S, P(x, q_1) = 1\}$. In the sequel, for the sake of simplicity, we will denote by $\mathbb{O}$ and $\mathbb{I}$ the equivalence classes $[\mathbb{O}]$ and $[\mathbb{I}]$, respectively. The elements of the quotient set $\mathcal{F} := Q \backslash \equiv_s$ will be called *effects*.

Having fixed an effect $f \in \mathcal{F}(\mathcal{H})$, we can associate with it the following subsets of S:

- The *certainly-yes domain* of f: $S_1(f) := \{x \in S: P(x, f) = 1\}$.
- The *certainly-no domain* of f: $S_0(f) := \{x \in S: P(x, f) = 0\}$.

Making use of the equivalence relation (1.2), it is easy to prove, from Axioms 1–6, that the following theorem holds:

Theorem 1.1. Let (S, Q, P) be a state–question–probability structure. Defining the *probability function* $S \times \mathscr{F} \mapsto [0, 1]$, for simplicity also denoted by P, for every $x \in S$ and every $f \in \mathscr{F}$, by

$$P(x, f) := P(x, q_f), \qquad q_f \in f \tag{1.3}$$

the induced state–effect–probability triple $(S, \mathscr{F}, P)$ satisfies the following conditions:

(SEP-1) The existence of the *certain* and the *impossible effect*:

$$\exists \mathbb{I} \in \mathscr{F}: \quad \forall x \in S, \quad P(x, \mathbb{I}) = 1$$
$$\text{and} \qquad \exists \mathbb{O} \in \mathscr{F}: \quad \forall x \in S, \quad P(x, \mathbb{O}) = 0$$

(SEP-2) The *indistinguishability principle of states*: $\forall f \in \mathscr{F}$, $P(x_1, f) = P(x_2, f) \Rightarrow x_1 = x_2$.

(SEP-3) The *indistinguishability principle of effects*: $\forall x \in S$, $P(x, f_1) = P(x, f_2) \Rightarrow f_1 = f_2$.

(SEP-4) The existence of the *inverse effect*. For every effect $f = [q] \in \mathscr{F}$, we can introduce the *inverse effect* of f as the equivalence class of questions $f' := [q']$ [which, owing to (qK-0), is well defined] such that $\forall x \in S$, $P(x, f) + P(x, f') = 1$.

(SEP-5) The *partial sum operation* for pairs of orthogonal effects. For every pair of effects $f, g \in \mathscr{F}$ satisfying the following *orthogonality condition* $\forall x \in S$, $P(x, f) + P(x, g) \leq 1$, there exists an effect, denoted by $f \oplus g$ and called the *sum* of f and g, such that

$$\forall x \in S, \quad P(x, f \oplus g) = P(x, f) + P(x, g)$$

(SEP-6) For every number $\lambda \in [0, 1]$ and every effect $f \in \mathscr{F}$, there exists an effect, denoted by λf, such that for every $x \in \mathscr{S}$, $P(x, \lambda f) = \lambda P(x, f)$.

(SEP-7) For every effect $f \in \mathscr{F}$, an effect $f^{\nu} \in \mathscr{F}$ exists (the *necessity* of f) such that
 (i) $S_1(f^{\nu}) = S_1(f)$;
 (ii) If $g \in \mathscr{F}$ satisfies $S_1(g) = S_1(f)$, then $\forall x \in S$, $P(x, f^{\nu}) \leq P(x, g)$.
 (iii) If $h \in \mathscr{F}$ satisfies $S_1(h) = S_1((f^{\nu})')$, then $\forall x \in S$, $P(x,$

$$(f^{\nu})') \leq P(x, h).$$

The “orthodox” version of Axiom 5 in ref. 7 immediately follows from (SEP-5) and (SEP-6):

Axiom 5-CGN. For every finite family of effects $\{f_1, \ldots, f_n\} \subseteq \mathcal{F}$ and every corresponding finite family of nonnegative real numbers $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{R}_+$ such that $\Sigma_{j=1}^{n} \lambda_j = 1$, an effect $\Sigma_{j=1}^{n} \lambda_j f_j \in \mathcal{F}$ exists [the *convex combination* of the f_j with *weights* λ_j] such that

$$\forall x \in \mathcal{S}, \quad P\left(x, \sum_{j=1}^{n} \lambda_j f_j\right) = \sum_{j=1}^{n} \lambda_j P(x, f_j)$$

Thus any state-effect-probability structure has the *convex property*. In particular, for any finite set of effects $f_1, f_2, \ldots, f_n$ we denote by πf_j the *product effect*, i.e., the equiweighted convex combination ($\forall j$, $\lambda_j = 1/n$). The product effect of two effects f and g will be denoted also by $f \cdot g$. Trivially, for any convex combination with $\lambda_j \neq 0$, $j = 1, \ldots, n$, (and so for any product) of effects we have

$$S_1\left(\sum_{j=1}^{n} \lambda_j f_j\right) = \bigcap_{j=1}^{n} S_1(f_j) \qquad \text{and} \qquad S_0\left(\sum_{j=1}^{n} \lambda_j f_j\right) = \bigcap_{j=1}^{n} S_0(f_j) \tag{1.4}$$

As shown in ref. 9, the set of effects of an abstract triple satisfying (SEP-1)–(SEP-7) forms a *BZ effect algebra* $\langle \mathcal{F}, \oplus, ', {}^{\sim}, \mathbb{O}, \mathbb{I} \rangle$ under the partial sum operation (SEP-5), the *Kleene complement* (SEP-4), and the *Brouwer complement* $\forall f \in \mathcal{F}, f^{\sim} := f^{'\nu}$ defined by (SEP-7) and (SEP-4). The standard partial order relation induced in every effect algebra $f \leq g \Leftrightarrow_{def} \exists h \in \mathcal{F}$: $f \oplus h = g$ coincides in this case with the *physical partial order* $\forall x \in S$, $P(x, f) \leq P(x, g)$.

Recall that, in particular, the *Brouwer complementation* satisfies the *weak double negation law*: $\forall f \in \mathcal{F}, f \leq f^{\sim\sim}$. This makes it possible to single out the set of B-*sharp* elements $\mathcal{E} := \{a \in \mathcal{F}: a = a^{\sim\sim}\}$ in which the two complementations collapse in a unique standard orthocomplementation ($\forall a \in \mathcal{E}$, $a' = a^{\sim}$) giving rise to the structure of sharp quantum logic (i.e., an orthomodular lattice).

1.3. Propositions and Properties

We now introduce, in the state–effect–probability structure $(S, \mathcal{F}, P)$, the equivalence relation

$$f_1 \equiv_p f_2 \Leftrightarrow_{def} S_1(f_1) = S_1(f_2) \Leftrightarrow (f_1)^{\nu} = (f_2)^{\nu} \tag{1.5}$$

Denote by $[f]_{(p)}$ the equivalence class generated by the effect $f \in \mathcal{F}$, and by

$\mathcal{E}_{(p)} = \mathcal{F} \backslash \equiv_p$ the quotient set of effects with respect to this equivalence relation. In particular, $[\mathbb{I}]_{(p)} = \{\mathbb{I}\}$ and $[\mathbb{O}]_{(p)} = \{\hat{f} \in \mathcal{F}: \forall x \in S, P(x, \hat{f}) \neq 1\}$. Following Piron [16], the elements of $\mathcal{E}_{(p)}$ will be called potentially measurable *propositions* of the system. From (1.5) we have that the following one-to-one correspondence identifying propositions and events:

$$[f]_{(p)} \in \mathcal{E}_{(p)} \leftrightarrow \mathcal{E} \ni f^{\nu} \tag{1.6}$$

With every proposition $\mathbf{a} \in \mathcal{E}_{(p)}$ we can associate the certainly-yes domain $S_1(\mathbf{a})$ defined as $S_1(f)$ for $f \in \mathbf{a}$. It is not possible, on the other hand, to associate a unique certainly-no domain with propositions. Property (SEP-7) of Theorem 1.1 can be restated in the following way:

(SEP-7p). For every proposition $\mathbf{a} \in \mathcal{E}_{(p)}$, there exists an event (B-sharp effect) $a \in \mathcal{E}$ such that:
1. $S_1(a) = S_1(\mathbf{a})$.
2. $\forall g \in \mathbf{a}, a \leq g$.
3. $\forall h \in [a']_{(p)}, a' \leq h$.

Note that 1 and 2 imply that the event a associated with proposition $\mathbf{a}$ will be unique. Furthermore, condition 3 ensures that a is the event associated with the proposition $[a]_{(p)}$ with certainly-yes domain $S_1(a)$ if and only if a' is the event associated with the proposition $[a']_{(p)}$ with certainly-yes domain $S_0(a)$.

In general, many properties of the physical entity described by a state–effect–probability structure are associated with every proposition $\mathbf{a}$; we can say that any effect $f \in \mathbf{a}$ "tests" all the properties associated with proposition $\mathbf{a}$. The event $f^{\nu} \in \mathcal{E}$ belongs to proposition $[f]_{(p)}$, and so it reveals those samples of the physical entity which possess all the properties associated with $[f]_{(p)}$. It is the most selective among all the measurements available in $[f]_{(p)}$ (i.e., $\hat{f} \in [f]_{(p)}$ implies $f^{\nu} \leq \hat{f}$); other elements of $[f]_{(p)}$ can be considered to be fuzzy representations of this proposition. Furthermore, it does not reveal, as much as possible, the ones which do not possess these properties, i.e., f^{ν} has the greatest certainly-no domain with respect to all the other effects in the same proposition:

$$\cup\{S_0(f): f \in [f]^{(p)}\} = S_0(f^{\nu})$$

and so "it minimizes the randomness of the 'no' answers" [13].

Note that, if we introduce on $\mathcal{F}$ the equivalence relation defined by

$$f_2 \equiv_n f_2 \Leftrightarrow_{def} S_0(f_1) = S_0(f_2) \Leftrightarrow (f_1)^{\sim} = (f_2)^{\sim} \tag{1.7}$$

we can denote by $[f]_{(n)}$ the equivalence class generated by f and by $\mathcal{E}_{(n)}$ the quotient set $\mathcal{F} \backslash \equiv_n$. Elements $\tilde{a} \in \mathcal{E}_{(n)}$ are called 'nopositions', and from

(1.7) the following one-to-one correspondence identifying nopositions and events holds:

$$[f]_{(n)} \in \mathscr{E}_{(n)} \leftrightarrow \mathscr{E} \ni f^{\sim} \qquad (1.8)$$

Then it is easy to prove the following proposition [10]:

Proposition 1.1. a is an event corresponding to the proposition $[a]_{(p)}$ with certainly-yes domain $S_1(a)$ if and only if a is a nonevent corresponding to the noposition $[a]_{(n)}$ with certainly-no domain $S_0(a)$.

This proposition guarantees that the set of propositions and the set of nopositions coincide: $\mathscr{E}_{(p)} = \mathscr{E}_{(n)}$; moreover, owing to the one-to-one correspondences (1.6) and (1.8), this set can be identified with the set $\mathscr{E}$ of all events.

2. UNSHARP LINEAR QUANTUM MECHANICS IN HILBERT SPACES

2.1. States, Effects, and Probability in Linear Quantum Mechanics on Hilbert Spaces

Let $\mathscr{H}$ be a complex, separable Hilbert space and let $\mathscr{H}' := \mathscr{H} \setminus \{\underline{0}\}$. We denote by $S(\mathscr{H})$ the set of one-dimensional subspaces of $\mathscr{H}$ except the zero vector, and by $x, y, \ldots$ elements of $S(\mathscr{H})$. We denote by $F_L(\mathscr{H})$ the set of *linear effects operators* (linear, bounded, self-adjoint, positive, and absorbing operators):

$$F_L(\mathscr{H}) := \{F \in \mathscr{L}(\mathscr{H}) \colon F = F^*, \forall \psi \in \mathscr{H}, 0 \le \langle \psi | F\psi \rangle \le \|\psi\|^2\}$$

The triple of states–effects–probability based on the Hilbert space $\mathscr{H}$ and satisfying conditions (SEP-1)–(SEP-6) of Theorem 1.1 [9] is given by $(S(\mathscr{H}), F_L(\mathscr{H}), P)$, where for any state $x \in \mathscr{S}(\mathscr{H})$ and any linear effect $F \in F_L(\mathscr{H})$, the *probability function* is defined by

$$P(x, F) := \frac{\langle \psi_x | F\psi_x \rangle}{\|\psi_x\|^2}, \qquad \psi_x \in x \qquad (2.1)$$

and is independent of the representative nonzero vector ψ_x chosen in x. In particular:

1. The *certain effect* is the identity operator $\mathbb{I}\colon \mathscr{H} \mapsto \mathscr{H}, \psi \to \mathbb{I}(\psi) := \psi$.
2. The *impossible effect* is the null operator $\mathbb{O}\colon \mathscr{H} \mapsto \mathscr{H}, \psi \to \mathbb{O}(\psi) := \mathbf{0}$.
3. For each $F \in F_L(\mathscr{H})$, the corresponding *inverse effect* is $F' := \mathbb{I} - F \in F_L(\mathscr{H})$.
4. Let $F, G, \in F_L(\mathscr{H})$ be linear effects on $\mathscr{H}$. Then, the *orthogonality*

condition $\forall \psi \in \mathcal{H}: \langle \psi | F\psi \rangle + \langle \psi | G\psi \rangle \leq 1$ implies that the algebraic sum of operators $F + G$ (the *partial sum* operation $F \oplus G := F + G \in F_L(\mathcal{H})$, defined for pairs of mutually orthogonal linear effects) is also an effect, too.

Recall that, for any pair of linear effects $F_1, F_2 \in F_L(\mathcal{H})$, the *physical indistinguishability* relation coincides with the identity relation on operators:

$$\forall x \in \mathcal{S}(\mathcal{H}), \quad P(x, F_1) = P(x, F_2) \Leftrightarrow F_1 = F_2 \tag{2.2}$$

Moreover, in the present case the physical partial order relation on linear effects assumes, for arbitrary $F_1, F_2 \in F_L(\mathcal{H})$, the form

$$F_1 \leq F_2 \Leftrightarrow \forall \psi \in \mathcal{H}, \quad \langle \psi | F_1 \psi \rangle \leq \langle \psi | F_2 \psi \rangle \tag{2.3}$$

With every linear effect $F \in F_L(\mathcal{H})$ we can associate its *certainly-yes domain* $\mathcal{S}_1(F) = \{x \in \mathcal{S}(\mathcal{H}): P(x, F) = 1\}$, which can be identified with the corresponding *certainly-yes subspace*:

$$M_1(F) := \{\psi_x \in \mathcal{H}': x \in \mathcal{S}_1(F)\} \cup \{0\} = \ker(\mathbb{I} - F) \tag{2.4}$$

and its *certainly-no domain* $\mathcal{S}_0(F) = \{x \in \mathcal{S}(\mathcal{H}): P(x, F) = 0\}$, which can be identified with the corresponding *certainly-no subspace*:

$$M_0(F) := \{\psi_x \in \mathcal{H}': x \in \mathcal{S}_0(F)\} \cup \{0\} = \ker(F) \tag{2.5}$$

The equivalence relation over the set of linear effects which defines properties on this Hilbert space framework is now

$$F_1 \equiv_p F_2 \Leftrightarrow \ker(\mathbb{I} - F_1) = \ker(\mathbb{I} - F_2) \tag{2.6}$$

Each $\equiv_p$ equivalence class $[F]_{(p)}$ corresponds to a *linear proposition* and contains a ‘representative’ sharp element which in the present linear case is the linear projection $\mathbb{P}_{M_1(F)}$ onto the certainly-yes subspace $M_1(F)$ generated, according to (2.4), by the common certainly-yes domain $\mathcal{S}_1(F)$. That is, the following theorem is easily proved.

Theorem 2.1. For each linear proposition $[F]_{(p)} \in \mathcal{E}_{(p)}(\mathcal{H})$ the linear event (B-sharp effect) $\mathbb{P}_{M_1(F)}$ satisfies condition (SEP-7p):

1. It is the B-sharp linear effect measuring the proposition since $\mathbb{P}_{M_1(F)} \in [F]_{(p)}$.
2. Every other linear effect operator $G \in [F]_{(p)}$ is a fuzzy (unsharp) representation of the proposition $[F]_{(p)}$ since $\mathbb{P}_{M_1(F)} \leq G$.
3. For every linear effect operator $\hat{F} \in [\mathbb{P}_{M_1(F)^\perp}]_{(p)}$ we have that $\mathbb{P}_{M_1(F)^\perp} \leq \hat{F}$.

The set $\Pi(\mathcal{H})$ of all linear projectors for $\mathcal{H}$ is the set of all exact (B-sharp) linear events and is an orthocomplemented orthomodular (not Boolean) lattice, i.e., a quantum logic, identified with the set of all linear propositions:

$$[F]_{(p)} \in \mathscr{E}_{(p)}(\mathscr{H}) \leftrightarrow \Pi(\mathscr{H}) \ni \mathbb{P}_{M_1(F)} \tag{2.7}$$

Note that every $P \in \Pi(\mathscr{H})$ is then linear, bounded, self-adjoint, and idempotent.

Example 2.1. Consider the Hilbert space $\mathbb{C}^2$. The linear operator

$$F_{NM} := \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{2.8}$$

is a linear effect operator such that

$$\langle(\psi_r, \psi_v)|F_{NM}(\psi_r, \psi_v)\rangle = |\psi_r|^2 + \frac{1}{2}|\psi_v|^2 \tag{2.9}$$

The certainly-yes subspace of (2.8) is $\mathbb{C}_r^2 = \{(\psi_r, \psi_v) \in \mathbb{C}^2: \psi_v = 0\}$, whereas the certainly-no subspace consists of the single zero vector. The necessity of this linear effect is then the linear projection

$$(F_{NM})^\nu := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{P}_{\mathbb{C}_r^2} \tag{2.10}$$

and so

$$\langle(\psi_r, \psi_v)|(F_{NM})^\nu (\psi_r, \psi_v)\rangle = |\psi_r|^2 \tag{2.11}$$

The proposition generated by F_{NM} contains, for instance, the linear effects of the form

$$F_k := \begin{pmatrix} 1 & 0 \\ 0 & j/k \end{pmatrix} \tag{2.12}$$

with j, k any real numbers >1 such that $k > j$.

3. UNSHARP NONLINEAR QUANTUM MECHANICS IN HILBERT SPACES

3.1. From Linear to Nonlinear Quantum Mechanics on Hilbert Spaces

Let $\mathscr{H}$ be a (complex, separable) Hilbert space; the nonlinear approach consists of the extension of the class of linear effect operators $F_L(\mathscr{H})$ to a larger class $Q(\mathscr{H})$ of not-necessarily linear operators: in general, for an operator F: $\mathscr{H} \mapsto \mathscr{H}$ of this class the additivity condition $\forall\varphi, \psi \in \mathscr{H}$, $F(\varphi + \psi) = F(\varphi) + F(\psi)$ is not required to hold.

Let F: $\mathcal{H} \to \mathcal{H}$ be a homogeneous operator [i.e., such that $\forall\alpha \in \mathbb{C}$, $\forall\psi \in \mathcal{H}$: $F(\alpha\psi) = \alpha F(\psi)$]. Consider the set $N(\mathcal{H})$ of homogeneous operators which are bounded, i.e., $sup\{\|F\psi\|: \|\psi\| = 1\} < +\infty$. The mapping $\|\cdot\|$: $N(\mathcal{H}) \mapsto \mathbb{R}_+$, $F \to \|F\| := sup\{\|F\psi\|: \|\psi\| = 1\}$ is a norm on $N(\mathcal{H})$, namely:

1. $\|F\| = 0 \Leftrightarrow F = \mathbb{O}$.
2. $\forall\lambda \in \mathbb{C}$, $\|\lambda F\| = |\lambda| \cdot \|F\|$.
3. $\|F_1 + F_2\| \leq \|F_1\| + \|F_2\|$.

It is immediately verified that each $F \in H(\mathcal{H})$ is continuous at zero, but in general not on the whole of $\mathcal{H}$. Furthermore, if F is not linear, the adjoint F^* defined over all of $\mathcal{H}$ in general does not exist.

Example 3.1. Let $\{u_n\}$ be an orthonormal basis in $\mathcal{H}$. The operator F: $\mathcal{H} \to \mathcal{H}$ defined by

$$F\psi := \begin{cases} 2\alpha u_2 & \text{if} \quad \psi = \alpha u_1, \ \alpha \in \mathbb{C} \\ \psi & \text{if} \quad \psi \neq \alpha u_1, \ \alpha \in \mathbb{C} \end{cases}$$

is homogeneous, nonadditive [$F(u_1 + u_2) = u_1 + u_2$, $F(u_1) = 2u_2$, $F(u_2) = u_2$], and such that $\forall\psi \in \mathcal{H}$, $\mathbb{O} \leq \langle\psi|F\psi\rangle \leq \|\psi\|^2$, but $\|F\| = 2$. Moreover, $\langle Fu_2|u_1\rangle = 0$ and $\langle u_2|Fu_1\rangle = 2$, i.e., $\langle Fu_2|u_1\rangle \neq \langle u_2|Fu_1\rangle$ [this operator is not self-adjoint], whereas $\forall\psi \in \mathcal{H}$, $\langle\psi|F\psi\rangle = \langle F\psi|\psi\rangle$, i.e., it is *diagonally* self-adjoint.

We denote by $Q(\mathcal{H})$ the set of homogeneous operators $F \in N(\mathcal{H})$ which are bounded by the identity operator $\mathbb{I}$, diagonally self-adjoint, positive, and absorbing. Namely, $F \in Q(\mathcal{H})$ if and only if:

1. $\forall\alpha \in \mathbb{C}$, $\forall\psi \in \mathcal{H}$: $F(\alpha\psi) = \alpha F(\psi)$.
2. $\|F\| \leq \|\mathbb{I}\| = 1$.
3. $\forall\psi \in \mathcal{H}$, $\langle F\psi|\psi\rangle = \langle\psi|F\psi\rangle$ [which implies $\forall\psi \in \mathcal{H}$, $\langle\psi|F\psi\rangle \in \mathbb{R}$].
4. $\forall\psi \in \mathcal{H}$, $\mathbb{O} \leq \langle\psi|F\psi\rangle \leq \|\psi\|^2$.

In the following, elements from $Q(\mathcal{H})$ will be called *homogeneous* (sometimes also *effect*) *operators*. The probability (2.1) defined above can be extended to the new class of homogeneous operators as a well-defined quantity in the real unit interval, which is independent of the representative nonzero vector ψ_x in the ray x. But in this homogeneous case, the identity condition (2.2) in general does not hold.

Example 3.2. The operator F: $\mathbb{C}^3 \to \mathbb{C}^3$ defined as

$$F(z_1, z_2, z_3) := \begin{cases} (0, 0, 0) & \text{if} \quad z_3 = 0 \\ (z_1\overline{z_2}/\overline{z_3}, \ -z_1\overline{z_2}/\overline{z_3}, \ 0) & \text{if} \quad z_3 \neq 0 \end{cases}$$

is a homogeneous effect operator such that $F \neq \mathbb{O}$, but $\langle\psi|F\psi\rangle = 0$ for all $\psi \in \mathbb{C}^3$.

As described in Section 1.2, in this homogeneous extension it is meaningful to introduce the equivalence relation of physical indistinguishability:

$$F_1 \equiv_S F_2 \Leftrightarrow_{def} \forall x \in \mathcal{S}(\mathcal{H}), \quad P(x, F_1) = P(x, F_2) \tag{3.1a}$$

$$\Leftrightarrow \forall \psi \in \mathcal{H}, \quad \langle \psi | F_1 \psi \rangle = \langle \psi | F_2 \psi \rangle \tag{3.1b}$$

Each $\equiv_S$ equivalence class defines an *effect* and we denote by $\mathcal{F}(\mathcal{H})$ the collection of all such effects $f = [F]_S$, where $[F]_S$ is as usual the equivalence class of physically indistinguishable effect operators picked out by the homogeneous effect operator F.

The following result is quite trivial.

Proposition 3.1. Let $f \in \mathcal{F}(\mathcal{H})$ be an effect. If there exists a linear effect operator $F_l \in f$, then this operator is unique:

$$\forall F_l, G_l \in F_L(\mathcal{H}), \quad F_l, G_l \in f \Rightarrow F_l = G_l \tag{3.2}$$

We can then divide all effects into two classes: the effects generated by a (unique) linear effect operator, also called *linear effects* and whose collection is denoted by $\mathcal{F}_L(\mathcal{H})$, and effects which are purely nonlinear. Of course, from Proposition 3.1 we have that the set of all linear effects can be identified with the set of all linear effect operators by the one-to-one correspondence $F_L(\mathcal{H}) \leftrightarrow \mathcal{F}_L(\mathcal{H})$, associating with every linear effect operator $F_l \in \mathcal{F}_L(\mathcal{H})$ the linear effect (equivalence class) $[F_l]_S$.

Example 3.3. Taking into account the linear effect operator (2.8) of Example 2.1, the following homogeneous effect operator is physically indistinguishable from it:

$$\hat{F}_{NM}(\psi_r,\psi_v) = \begin{cases} \left(\psi_r + \dfrac{|\psi_v|^2}{2|\psi_r|^2}\,\psi_r,\ 0\right) & \text{if} \quad \psi_r \neq 0 \\ \left(0,\ \tfrac{1}{2}\psi_v\right) & \text{if} \quad \psi_r = 0 \end{cases} \tag{3.3}$$

Indeed, for every $(\psi_r, \psi_v) \in \mathbb{C}^2$ we have that $\langle(\psi_r, \psi_v)|\hat{F}_{NM}(\psi_r, \psi_v)\rangle$ is equal to (2.9).

Example 3.4. Define on $\mathbb{C}^2$ the homogeneous effect operator

$$F_{GP}(\psi_r,\psi_v) := \begin{cases} \left(\psi_r + \dfrac{1}{2}\dfrac{|\psi_v|^2}{|\psi_r|^2}\,\psi_r,\ 0\right) & \text{if} \quad \psi_r \neq 0 \\ (0,\ 0) & \text{if} \quad \psi_r = 0 \end{cases} \tag{3.4}$$

This operator is idempotent and generates an effect which is purely nonlinear (there is no linear effect operator physically indistinguishable from it).

3.2. The Hilbert Homogeneous Model of the State–Effect–Probability Structure

Define the map P: $S(\mathcal{H}) \times \mathcal{F}(H) \to [0, 1]$ by

$$P(x, f) := \frac{\langle \psi_x | F_f \, \psi_x \rangle}{\|\psi_x\|^2}, \qquad \psi_x \in x, \quad F_f \in f$$

We have constructed a triple $(S(\mathcal{H}), \mathcal{F}(\mathcal{H}), P)$ which satisfies the following conditions:

(SEP-1) The certain effect (given by the equivalence class of the identity operator $\mathbb{I}$: $\mathcal{H} \to \mathcal{H}$) and the impossible effect (given by the equivalence class of the null operator $\mathbb{O}$: $\mathcal{H} \to \mathcal{H}$) both exist.

(SEP-2) The principle of indistinguishability of states is clearly satisfied: if $x_1 \neq x_2$, the nonlinear operator F defined by

$$F\psi := \begin{cases} \psi & \text{if} \quad \psi \in x_1 \\ \frac{1}{2}\psi & \text{if} \quad \psi \in x_2 \\ \mathbf{0} & \text{if} \quad \psi \notin x_1 \cup x_2 \end{cases}$$

is an effect operator and, denoting by f_F the effect generated by this effect operator, $P(x_1, f_F) = 1$, $P(x_2, f_F) = 1/2$.

(SEP-3) The principle of indistinguishability of effects is also satisfied: in fact, if $P(x, f_1) = P(x, f_2)$ for all $x \in S(\mathcal{H})$, then for every $F_1 \in f_1$ and every $F_2 \in f_2$, $\langle \psi | F_1 \psi \rangle = \langle \psi | F_2 \psi \rangle$ for all $\psi \in \mathcal{H}$, that is, $f_1 = f_2$ (as equivalence classes).

(SEP-4) The existence of the inverse is satisfied: for every $F \in Q(\mathcal{H})$ set, as usual, $F' := \mathbb{I} - F$; if $F_1 \equiv_S F_2$ $[\forall \psi \in \mathcal{H}, \langle \psi | F_1 \psi \rangle = \langle \psi | F_2 \psi \rangle]$, then $F_1' \equiv_S F_2'$ $[\forall \psi \in \mathcal{H}, \langle \psi | (\mathbb{I} - F_1)\psi \rangle = \langle \psi | (\mathbb{I} - F_2)\psi \rangle]$. Thus for every effect $f = [F]$ the effect $f' = [F']$ is well defined and independent of the representative operator $\hat{F} \in f$. Clearly, $P(x, f) + P(x, f') = 1$ for all $x \in S(\mathcal{H})$.

(SEP-5) The existence of the partial sum operation: if $F_1 \equiv_S \hat{F}_1$ and $F_2 \equiv_S \hat{F}_2$ with $\langle \psi | F_1 \, \psi \rangle + \langle \psi | F_2 \psi \rangle \leq 1$ for every $\psi \in \mathcal{H}$, then $\langle \psi | (F_1 + F_2)\psi \rangle = \langle \psi | (\hat{F}_1 + \hat{F}_2)\psi \rangle \leq 1$, i.e., the sum operation $[F_1]_S + [F_2]_S := [F_1 + F_2]_S$ is well defined, and is an effect.

(SEP-6) The existence of the convex product: if $F \equiv_S G$, then for every $\lambda \in [0, 1]$ we have that $\langle \psi | (\lambda F)\psi \rangle = \langle \psi | (\lambda G)\psi \rangle$, i.e., the product $\lambda[F] := [\lambda F]$ is well defined.

The partial order relation for $\mathcal{F}(\mathcal{H})$ in the present homogeneous context becomes

$$f_1 \leq f_2 \Leftrightarrow \forall \psi \in \mathcal{H}, \quad \langle \psi | F_1 \, \psi \rangle \leq \langle \psi | F_2 \psi \rangle \quad \text{where} \quad F_1 \in f_1, \quad f_2 \in f_2$$

Having fixed an effect $f \in \mathcal{F}(\mathcal{H})$, we have:

- The *certainly-yes* domain of f, $S_1(f) := \{x \in S(\mathcal{H}) \colon P(x, f) = 1\}$, can be identified with the subset of $\mathcal{H}$: $M_1(F) := \{\psi \in \mathcal{H} \colon \langle \psi | F\psi \rangle = \langle \psi | \psi \rangle\}$, independent of the representative $F \in f$ and sometimes denoted also by $M_1(f)$.
- The *certainly-no* domain of f, $S_0(f) := \{x \in S(\mathcal{H}) \colon P(x, f) = 0\}$, can be identified with the subset of $\mathcal{H}$: $M_0(F) := \{\psi \in \mathcal{H} \colon \langle \psi | F\psi \rangle = 0\}$, independent of the representative $F \in f$ and sometimes denoted also by $M_0(f)$.

It is easy to establish the following characterization of certainly-yes and -no domains:

$$M_1(F) = \bigcup_{G \in [F]_S} \ker(G') \qquad \text{and} \qquad M_0(F) = \bigcup_{G \in [F]_S} \ker(G) \tag{3.5}$$

where $\ker(G) := \{\psi \; \mathcal{H} \colon G\psi = 0\}$ is not, in general, a subspace of $\mathcal{H}$.

To prove the first equation in (3.5), note that obviously $\cup_{G \in [F]_S} \ker(G') \subseteq M_1(F)$. On the other hand, given a $\psi_0 \in M_1(F)$, let x_0 be the one-dimensional subspace generated by ψ_0 and define the operator $G_0 \colon \mathcal{H} \to \mathcal{H}$,

$$G_0 \psi := \begin{cases} \psi & \text{if} \quad \psi \in x_0 \\ F\psi & \text{if} \quad \psi \notin x_0 \end{cases}$$

This operator is homogeneous with $G_0 \in [F]_S$ and $G_0 \psi_0 = \psi_0$, so that $S_1(F) \subseteq \cup_{G \in [F]_S} \ker(G')$. The second equation in (3.5) is proved in analogous fashion.

In the standard Hilbert space structure of quantum mechanics, effects are given by linear, self-adjoint, positive operators bounded by the identity operator $\mathbb{I}$, and the respective certainly-yes and -no domains are the *subspaces* (2.4) and (2.5) of the Hilbert space $\mathcal{H}$. In the present approach using homogeneous operators it is clear from (3.5) that the certainly-yes and -no domains are given by *set-theoretic unions of one-dimensional subspaces*, which from now on we will call *starred subsets* and whose collection will be denoted by $\Sigma(\mathcal{H})$.

In fact, if we take x_ψ to be the one-dimensional subspace generated by $\psi \in \mathcal{H}$, from (3.5) we have that

$$M_1(F) = \bigcup_{\substack{G \in [F]_S \\ \psi \in \ker(G')}} x_\psi \qquad \text{and} \qquad M_0(F) = \bigcup_{\substack{G \in [F]_S \\ \psi \in \ker(G)}} x_\psi \tag{3.6}$$

Consider now the starred subset $M \in \Sigma(\mathscr{H})$, and define the operator $\chi_M: \mathscr{H} \mapsto \mathscr{H}$ in the following way:

$$\forall \psi \in \mathscr{H}, \qquad \chi_M(\psi) := \begin{cases} \psi & \text{if} \quad \psi \in M \\ \mathbf{O} & \text{if} \quad \psi \notin M \end{cases} \tag{3.7}$$

Because of the hypothesis that $M \in \Sigma(\mathscr{H})$, the map χ_M is clearly a homogeneous effect operator such that $M_1(\chi_M) = M$ and $M_0(\chi_M) = M^c$ (the set-theoretic complement $\mathscr{H} \backslash M$ of M).

We are now in a position to prove the following proposition:

Proposition 3.2. The homogeneous Hilbert model $(S(\mathscr{H}), \mathscr{F}(\mathscr{H}), P)$ satisfies condition (SEP-7) and therefore is a state–effect–probability structure. To be precise, for all $F \in Q(\mathscr{H})$, there exists an $F^\nu = \chi_{M_1(F)} \in Q(\mathscr{H})$ (the homogeneous necessity of F) such that:

1. $M_1(F) = M_1(\chi_{M_1(F)})$.
2. If $G \in \mathscr{F}(\mathscr{H})$ is such that $M_1(G) = M_1(\chi_{M_1(F)})$, then for every $\psi \in \mathscr{H}$, $\langle \psi | \chi_{M_1(F)} \psi \rangle \leq \langle \psi | G \psi \rangle$.
3. If $H \in \mathscr{F}(\mathscr{H})$ is such that $M_1(H) = M_1(\chi'_{M_1(F)})$, then for every $\psi \in \mathscr{H}$, $\langle \psi | \chi'_{M_1(F)} \psi \rangle \leq \langle \psi | H \psi \rangle$.

Proof. (1) is obvious. As for (2), we need to show that $\langle \psi | \chi_{M_1(F)} \psi \rangle \leq \langle \chi | G \psi \rangle$ for all $\psi \in \mathscr{H}$ if $M_1(G) = M_1(F)$. This is immediate both for $\psi \in M_1(F)$ and for $\psi \notin M_1(F)$. We now show (3). Let

$$M_1(H) = M_1(\chi'_{M_1(F)}) = M_0(\chi_{M_1(F)}) = (M_1(\chi_{M_1(F)}))^c$$

Then, if $\psi \in \chi_{M_1(F)}$, it follows that $\chi'_{M_1(F)}\psi = 0$. If, on the other hand, $\psi \in (M_1(\chi_{M_1(F)}))^c = M_1(H)$, then $\langle \psi | \chi'_{M_1(F)} \psi \rangle \leq \langle \psi | H \psi \rangle$, so that, in any case, $\langle \psi | \chi'_{M_1(F)} \psi \rangle \leq \langle \psi | H \psi \rangle$. ■

If, for the sake of simplicity, we identify any Hilbertian effect $[F]_S \in \mathscr{F}(\mathscr{H})$ (as equivalence class of physically indistinguishable homogeneous effect operators) with any of its representative operators $F \in Q(\mathscr{H})$, then the Brouwerian complement turns out to be $F^\sim = (F')^\nu = \chi_{M_1(F')} = \chi_{M_0(F)}$, from which we have that $F^{\sim\sim} = \chi_{M_0(F)^c}$. Thus, the set of all Hilbertian B-sharp effects $\mathscr{E}(\mathscr{H}) = \{F \in Q(\mathscr{H}) \colon F = F^{\sim\sim}\}$ coincides with the collection of all homogeneous effects defined by (3.7):

$$\mathscr{E}(\mathscr{H}) = \{\chi_M \colon M \in \Sigma(\mathscr{H})\} \tag{3.8}$$

3.3. The Unsharp BZMV Classical Logic of Homogeneous Effects in the Hilbert Space Model

For every (linear or homogeneous) effect operator $F \in Q(\mathcal{H})$, construct the fuzzy set on the universe $\mathcal{H}'$, called the *probability distribution* of F, as the map φ_F: $\mathcal{H}' \mapsto [0, 1]$ defined by the law

$$\forall \psi \in \mathcal{H}', \quad \varphi_F(\psi) := \frac{\langle \psi | F \psi \rangle}{\|\psi\|^2} \tag{3.9}$$

which is a homogeneous function of degree zero, that is, $\varphi_F(\alpha\psi) = \varphi_F(\psi)$ for all $\alpha \in \mathbb{C}$ and all $\psi \in \mathcal{H}'$. Denote by Θ the collection of all functions (fuzzy sets on $\mathcal{H}'$) φ: $\mathcal{H}' \to [0, 1]$ which are homogeneous of degree zero.

With every effect operator $F \in Q(\mathcal{H})$ it is possible to associate the nonlinear (homogeneous, nonadditive) 'representative' $F^{(nl)}$: $\mathcal{H} \mapsto \mathcal{H}$ defined as

$$F^{(nl)}(\psi) := \begin{cases} \varphi_F(\psi)\psi & \psi \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

Trivially, the two effect operators F and $F^{(nl)}$ are physically indistinguishable: $\forall F \in Q(\mathcal{H})$, $F \equiv_S F^{(nl)}$, i.e., they generate the same effect. If $F_1, F_2 \in Q(\mathcal{H})$ are physically indistinguishable, then they generate the same nonlinear representative $F_1^{(nl)} = F_2^{(nl)}$. Therefore, for every effect $f \in \mathcal{F}$, we will refer to the operator $F^{(nl)}$, with $F \in f$, as the *canonical representative* of f.

Example 3.5. Referring to the linear effect operators (2.8) and (2.10) of Example 2.1, we have the following two nonlinear representatives, respectively:

$$F_{NM}^{(nl)}(\psi_r, \psi_v) = \begin{cases} \dfrac{|\psi_r|^2 + (1/2)|\psi_v|^2}{|\psi_r|^2 + |\psi_v|^2}(\psi_r, \psi_v) & \text{if} \quad (\psi_r, \psi_v) \neq 0 \\ (0, 0) & \text{if} \quad \psi_r = \psi_v = 0 \end{cases} \tag{3.11}$$

$$(F_{NM}^{v})^{(nl)}(\psi_r, \psi_v) = \begin{cases} \dfrac{|\psi_r|^2}{|\psi_r|^2 + |\psi_v|^2}(\psi_r, \psi_v) & \text{if} \quad (\psi_r, \psi_v) \neq 0 \\ (0, 0) & \text{if} \quad \psi_r = \psi_v = 0 \end{cases} \tag{3.12}$$

The homogeneous effect operator (3.11) is also the representative of (3.3), Example 3.3. The homogeneous effect operator (3.4), Example 3.4, is represented by

$$F_{GP}^{(nl)}(\psi_r, \psi_v) = \begin{cases} \dfrac{|\psi_r|^2 + (1/2)|\psi_v|^2}{|\psi_r|^2 + |\psi_v|^2}(\psi_r, \psi_v) & \text{if} \quad \psi_r \neq 0 \\ (0, 0) \quad \text{if} \quad \psi_r = 0 \end{cases} \tag{3.13}$$

Note that $F_{NM}(0, 1) = \hat{F}_{NM}(0, 1) = F_{NM}^{(nl)}(0, 1) = (0, 1/2)$, whereas

$F_{GP}(0, 1) = F_{GP}^{(nl)}(0, 1) = (0, 0)$; moreover, the linear effect f_{NM} generated by the physically indistinguishable effect operators F_{NM}, $\hat{F}_{NM}$, $F_{NM}^{(nl)}$ is different from the nonlinear effect f_{GP} generated by the physically indistinguishable, purely homogeneous effect operators F_{GP}, $F_{GP}^{(nl)}$ [indeed, $\langle(0, 1)|F_{NM}(0, 1)\rangle = 1/2$ and $\langle(0, 1)|F_{GP}(0, 1)\rangle = 0$].

But also if $(F_{NM}^{\nu})^{(nl)}(0, 1) = (0,0)$, we have that $(F_{NM}^{\nu})^{(nl)}(1, 1) = (1/2, 1/2)$ and $F_{GP}(1, 1) = (3/2, 0)$; in particular, the linear effect f_{NM}^{ν} generated by the physically indistinguishable effect operators F_{NM}^{ν}, $(F_{NM}^{\nu})^{(nl)}$ is different from the effects f_{NM} and f_{GP} [indeed, $\langle(1, 1)|(F_{NM}^{\nu})^{(nl)}(1, 1)\rangle = 1$ and $\langle(1, 1)\|F_{GP}(1, 1)\rangle = \langle(1, 1)|F_{NM}(1, 1)\rangle = 3/2$].

Definition 3.1. We will say that an effect $f \in \mathcal{F}(\mathcal{H})$ is a *homogeneous projector* iff its canonical representative is idempotent, i.e., iff $(F^{(nl)})^2 = F^{(nl)}$ (equivalently, iff $\varphi_F^2 = \varphi_F$) for arbitrary $F \in f$. The collection of all homogeneous projectors will be denoted by $\mathcal{E}(\mathcal{H})$.

In this nonlinear extension of unsharp quantum mechanics, the idempotency of a homogeneous effect operator does not guarantee that the associated effect is a projector.

Example 3.6. The homogeneous effect operator F_{GP} (3.4) of Example 3.4 is idempotent, but the canonical representative $F_{GP}^{(nl)}$, (3.13), is not idempotent [$F_{GP}^{(nl)}(1, 1) = 3/4(1, 1)$ and $(F_{GP}^{(nl)})^2(1, 1) = (3/4)^2(1, 1)$]. Therefore, the effect f_{GP} is not a projector.

But there is more: homogeneous projectors are characterized by (3.7), as shown in the following.

Proposition 3.3. An effect $f = [F]_s \in \mathcal{F}(\mathcal{H})$ is a homogeneous projector iff its canonical nonlinear representative $F^{(nl)}$ [see (3.10)] is equal to $\chi_{M_1(F)}$.

Proof. Sufficiency is obvious, so we prove only necessity. Let $\varphi_F^2 = \varphi_F$; then

$$\left(\frac{\langle\psi|F\psi\rangle}{\|\psi\|^2}\right)^2 = \frac{\langle\psi|F\psi\rangle}{\|\psi\|^2} \Rightarrow \frac{\langle\psi|F\psi\rangle}{\|\psi\|^2} = 0 \text{ or } 1$$

But $\langle\psi|F\psi\rangle = \|\psi\|^2$ if and only if $\psi \in M_1(F)$, and $\langle\psi|F\psi\rangle = 0$ if and only if $\psi \in M_0(F)$. Therefore, $\langle\psi|F\psi\rangle = \langle\psi|\chi_{M_1(F)}\psi\rangle$ for all $\psi \in \mathcal{H}$, from which we get that for every $\psi \in \mathcal{H}$

$$F^{(nl)}(\psi) = \begin{cases} \dfrac{\langle\psi|F\psi\rangle}{\|\psi\|^2}\psi & \text{if } \psi \neq \underline{0} \\ \underline{0} & \text{otherwise} \end{cases} = \begin{cases} \dfrac{\langle\psi|\chi_{M_1(F)}\psi\rangle}{\|\psi\|^2}\psi & \text{if } \psi \neq \underline{0} \\ \underline{0} & \text{otherwise} \end{cases}$$

from which [by (3.7)] the equality $F^{(nl)} = \chi_{M_1(F)}$ follows. ■

Since for every fuzzy set φ_F: $\mathcal{H}' \to [0, 1]$ from Θ, the above (3.10) defines a homogeneous effect operator, from what we have said up to now there clearly is a bijective correspondence between the set $\mathcal{F}(\mathcal{H})$ of all homogeneous effects and the set Θ of all fuzzy sets on $\mathcal{H}'$, according to which every effect $f \in \mathcal{F}(\mathcal{H})$ can be identified with $\varphi_f \in \Theta$, where φ_f is the fuzzy set φ_F with F arbitrarily chosen in f:

$$f \in \mathcal{F}(\mathcal{H}) \leftrightarrow \varphi_f := \varphi_F \in \Theta, \qquad \text{with} \quad F \in f \tag{3.14}$$

Note that Θ has the natural structure of a classical *BZMV algebra* $(\Theta, \oplus, ', {}^{\sim}, \Xi_{\emptyset}, \Xi_{\mathcal{H}'})$ of de Morgan type (for a definition see refs. 6 and 8) with respect to the pointwise operation of truncated sum $\forall \psi \in \mathcal{H}'$, $(\varphi_f \oplus \varphi_g)(\psi) := \min\{1, \varphi_f(\psi) + \varphi_g(\psi)\}$, the Kleene complement $\varphi_f'(\psi) := 1 - \varphi_f(\psi)$, the Brouwer complement $\varphi_f^{\sim}(\psi) := \Xi_{M_0(f)}(\psi)$ [the *characteristic functional* of $M_0(f)$, equal to 1 for $\psi \in M_0(f)$, and 0 otherwise]. Here, $\Xi_{\emptyset}$ (resp., $\Xi_{\mathcal{H}'}$) is the 0 (resp., 1) map on $\mathcal{H}'$, corresponding to the zero (resp., unit) element of the BZMV structure.

BZMV algebras are models of classical Lukasiewicz many-valued logics, with applications to standard fuzzy set theory [8]. Since the structure of a classical BZMV algebra can be translated to an analogous classical structure associated with $\mathcal{F}(\mathcal{H})$, we can conclude that the "logic" of nonlinear effects of the proposed Hilbert space model is the classical one of a BZMV algebra.

Through a slight modification of results proved in ref. 8 we will now show that the *sharp* part of the BZMV algebra Θ is the (classical) Boolean algebra of all characteristic functionals Ξ_M, with M ranging over the set of starred subsets. In other words, the sharp part of Θ is identifiable with the collection of all starred subsets $\Sigma(\mathcal{H})$.

3.4. The Classical Boolean Logic of Propositions in the Hilbert Homogeneous Model of the State–Effect–Probability Structure

Now consider the equivalence relation on $Q(\mathcal{H})$ defined by

$$F_1 \equiv_p F_2 \Leftrightarrow_{def} M_1(F_1) = M_1(F_2)$$

According to the general theory, if $F \in Q(\mathcal{H})$, we denote by $[F]_{(p)}$ the proposition (equivalence class) generated by F and by $\mathcal{E}_{(p)}(\mathcal{H}) := \mathcal{F}(\mathcal{H})/\sim$ the corresponding quotient set. To every proposition $\mathbf{a} \in \mathcal{E}_{(p)}(\mathcal{H})$ we can associate the certainly-yes domain $M_1(\mathbf{a})$ defined by the common certainly-yes domain $M_1(F)$, for any arbitrary $F \in \mathbf{a}$.

The following proposition holds.

Proposition 3.4. For every proposition $\mathbf{a} \in \mathscr{E}_{(p)}(\mathscr{H})$ the event (B-sharp effect) associated with $\mathbf{a}$ according to (SEP-7p) of Section 1.3 is the homogeneous projection $\chi_{M_1(\mathbf{a})}$.

Analogously, the equivalence relation (1.7) defining nopositions can be formulated in the following way:

$$F_1 \equiv_n F_2 \Leftrightarrow_{def} M_0(F_1) = M_0(F_2)$$

and the following result can be proved.

Proposition 3.5. For every noposition $\hat{\mathbf{a}} \in \mathscr{E}_{(n)}(\mathscr{H})$ the event (B-sharp effect) associated with $\hat{\mathbf{a}}$ is the homogeneous projection $\chi_{M_0(\hat{\mathbf{a}})}$.

Proof. Immediate from the previous proposition and proposition 1.1. ■

As pointed out at the end of Section 1.3, we have that $\mathscr{E}_{(p)}(\mathscr{H}) = \mathscr{E}_{(n)}(\mathscr{H})$, and according to (1.6), $\mathscr{E}_{(p)}(\mathscr{H})$ can be identified with the set $\mathscr{E}(\mathscr{H})$ of all homogeneous projectors. From (3.8) we deduce that the collection of all starred subsets $\Sigma(\mathscr{H})$ is identifiable with $\mathscr{E}(\mathscr{H})$ through the bijection

$$\chi: \quad \Sigma(\mathscr{H}) \mapsto \mathscr{E}(\mathscr{H}), \quad M \to \chi_M \tag{3.15}$$

We now show how $\Sigma(\mathscr{H})$ and $\mathscr{E}(\mathscr{H})$ can be identified from the point of view of their algebraic structure. We begin with the logic of starred subsets $(\Sigma(\mathscr{H}), \subseteq, {}^c, \emptyset)$, where $\subseteq$ is the usual set-theoretic inclusion and ^{c} the set-theoretic complement. Assuming conventionally that the empty set is the null one-dimensional subspace, $\Sigma(\mathscr{H})$ is a complete lattice since both the set-theoretic union and intersection of any family of starred subsets are starred subsets. Trivially, these two lattice operations of union and intersection are distributive. Finally, the map which associates with a starred subset its set-theoretic complement (which is a starred subset, too) is a standard orthocomplementation. Hence, $\Sigma(\mathscr{H})$ is a Boolean (complete) lattice, i.e., a *classic logic*.

The structure of a classical logic that is associated with $\Sigma(\mathscr{H})$ translates, then, to an analogous classical logic structure associated with $\mathscr{E}(\mathscr{H})$ by the bijection (3.15), so that the Boolean lattice $(\Sigma(\mathscr{H}), \subseteq, {}^c, \emptyset, \mathscr{H})$ is isomorphic to the Boolean lattice $(\mathscr{E}(\mathscr{H}), \leq, ', \mathbb{O}, \mathbb{I})$.

Summarizing, the logic of measurable properties on the system is a classical logic isomorphic to the logic of starred subsets. For every property M there exists a precise yes–no apparatus described by the nonlinear operator χ_M through which M is measured. Other elements of $[\chi_M]_{(p)}$ are experimental apparata which measure in a ‘fuzzy’ way the same property associated with χ_M.

Note, in conclusion, the following simple proposition:

Proposition 3.6. If there exists a linear projection $E \in \Pi(\mathscr{H})$ in the proposition $\mathbf{a}, \in \mathscr{E}(\mathscr{H})$, then $\chi_{M_1(\mathbf{a})} < E$.

Proof. Straightforward from the fact that $\forall_x \in S_p(\mathbf{a})$, $P(x, \chi_{S_p(\mathbf{a})}) = P(x, E) = 1$ and $\forall_X \notin S_p(\mathbf{a})$, $P(x, \chi_{S_p(\mathbf{a})}) = 0$. But there must exist at least one $x \notin S_p(\mathbf{a})$ such that $P(x, E) > 0$. ∎

4. CONCLUSIONS

This paper has shown how the nonlinear extension of quantum mechanics can be constructed as a triple $(S(\mathcal{H}), \mathcal{F}(\mathcal{H}), P)$, a state–effect–probability structure.

It has also shown that the set of all nonlinear effects $\mathcal{F}(\mathcal{H})$ has the algebraic structure of a BZMV-algebra, and therefore its logic is a classical fuzzy logic (Łukasiewicz many-valued logic). It (but not its sharp part) contains as a proper subset the set of standard quantum mechanical linear effects and projectors (linear propositions). The sharp part of $\mathcal{F}(\mathcal{H})$ has the structure of a Boolean algebra, and therefore its logic is a classical logic.

It is possible to say quite a bit more about this structure. We can prove that it is, in a precise sense, a classical extension of linear quantum mechanics: in fact, it is the *canonical classical extension* that Beltrametti and Bugajski defined and studied [1]. We will prove these results in a further paper.

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